Journal of Statistical Physics, Vol. 28, No. 3, 1982

Asymptotic Properties of Coupled Nonlinear Langevin Equations in the Limit of Weak Noise. II: Transition to a Limit Cycle

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Received November 6, 1981

We apply the singular perturbation technique, developed in the companion paper, to the study of the fluctuations at the onset of a limit cycle, both for the cases of a soft and a hard transition. The technique and results are illustrated on the Poincaré model (soft transition) and on the Van der Pol oscillator (hard transition).

KEY WORDS: Nonlinear stochastic differential equations; Fokker–Planck equations; fluctuations; Hopf bifurcation; limit cycle; hard transitions.

1. INTRODUCTION

The asymptotic properties of coupled Langevin equations in the limit of weak noise can be derived by a singular perturbation technique. In a previous paper, we applied such an analysis mainly to the case of a cusp bifurcation. In this paper we are concerned with the onset of a limit cycle. We consider both the case of a soft transition (Hopf bifurcation) and a hard one (like, for instance, the Van der Pol oscillator).

The main ideas of our approach can be summarized as follows (for a more detailed discussion see Ref. 1). Consider the set of Langevin equations:

$$\begin{aligned} \frac{d}{dt} x &= f(x, y) + \epsilon^{1/2} F_x \\ \frac{d}{dt} y &= g(x, y) + \epsilon^{1/2} F_y \end{aligned} \tag{1}$$

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0022-4715/82/0700-0577$03.00/0

where F_x and F_y represent Gaussian white noises, defined by

$$\begin{aligned} \langle F_x(t)F_x(t')\rangle &= Q_{xx}\delta(t-t') \\ \langle F_x(t)F_y(t')\rangle &= \langle F_y(t)F_x(t')\rangle = Q_{xy}\delta(t-t') \\ \langle F_y(t)F_y(t')\rangle &= Q_{yy}\delta(t-t') \end{aligned} \tag{2}$$

The problem is to study the asymptotic behavior of the stochastic process (x, y) in the limit $\epsilon \to 0$. One can show that for all finite times t:

$$\lim_{\epsilon\to 0}\begin{cases} |x(t)-\bar{x}(t)| = 0 \\ |y(t)-\bar{y}(t)| = 0 \end{cases} \tag{3}$$

with probability 1, where (x, y) obey the "macroscopic" equations:

$$\begin{aligned} \frac{d}{dt}\bar{x} &= f(\bar{x}, \bar{y}) \\ \frac{d}{dt}\bar{y} &= g(\bar{x}, \bar{y}) \end{aligned} \tag{4}$$

This result remains valid in the limit $t \to \infty$ if there is a unique and globally stable macroscopic stationary state (note that this includes the case of marginal stability). Let $(\bar{x}_s, \bar{y}_s)$ be this macroscopic stationary state:

$$f(\bar{x}_s, \bar{y}_s) = g(\bar{x}_s, \bar{y}_s) = 0 \tag{5}$$

By linearizing $(\bar{x}, \bar{y})$ around $(\bar{x}_s, \bar{y}_s)$, in Eqs. (4), we obtain

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x-\bar{x}_s \\ y-\bar{y}_s \end{bmatrix} &= \begin{bmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{bmatrix}\begin{bmatrix} x-\bar{x}_s \\ y-\bar{y}_s \end{bmatrix} \\ &= L\begin{bmatrix} x-\bar{x}_s \\ y-\bar{y}_s \end{bmatrix} \end{aligned} \tag{6}$$

The asymptotic behavior of the process (x, y) depends crucially on the spectral properties of the matrix L. Let λ_1 and λ_2 be its eigenvalues. In the neighbourhood of a cusp bifurcation, λ_1 and λ_2 are real and negative, one of them going to zero at the bifurcation point:

$$\lambda_1 < \lambda_2 \lesssim 0 \tag{7}$$

In this case L can always be diagonalized and hence, without loss of generality, one need only to consider the following equivalent problem:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \epsilon^{1/2}\begin{bmatrix} F_x \\ F_y \end{bmatrix} + \text{nonlinear terms} \tag{8}$$

The introduction of scaled variables (u, v),

$$\begin{aligned} u &= (x - \bar{x}_s)\epsilon^{-1+a} \\ v &= (y - \bar{y}_s)\epsilon^{-1+b}, \qquad a, b < 1 \end{aligned} \tag{9}$$

allows to derive the asymptotic properties of the process (u, v) before, near and at the bifurcation point. This was accomplished in Ref. 1.

In the case of transition to a limit cycle, the situation is entirely different since the two eigenvalues λ_1 and λ_2 are complex conjugate. The procedure should then be modified as follows.

Since at the bifurcation point the real part of λ_1 and λ_2 vanishes, the matrix L possesses purely imaginary eigenvalues. Hence, one can always apply a linear transformation which antidiagonalizes L. Therefore, instead of the equations (1) one can, without loss of generality, consider the following problem:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} R_1 & \omega \\ -\omega & R_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \epsilon^{1/2}\begin{bmatrix} F_x \\ F_y \end{bmatrix} + \text{nonlinear terms} \tag{10}$$

with R_1 and R_2 going to zero as the bifurcation point is approached. Note that in the case of a soft transition (Hopf bifurcation) we have $R_1 = R_2$, but this is not necessarily the case for a hard transition (see Section 3).

To study the process near or beyond the bifurcation, it is convenient to introduce polar coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{11}$$

Note that the latter transformation being nonlinear, one should take into account the subtleties of Ito–Stratonovich calculus.[2] For the classical calculation rules to be valid (Stratonovich interpretation), one has to add the so called "spurious drift" terms in the stochastic differential equations (10). Alternatively, one can always perform the change of variables (11) on the Fokker–Planck equation equivalent to (10), following the usual rules for partial differential equations. In either case, one obtains

$$\begin{aligned} \partial_t P(r, \theta; t) = &-\partial_r\Big[R_2 r - r\cos^2\theta(R_2 - R_1) + \cdots + \frac{\epsilon}{2r}\,Q_{\theta\theta}\Big]P \\ &- \partial_\theta\Big[-\omega + \sin\theta\cos\theta(R_2 - R_1) + \cdots - \frac{\epsilon}{r^2}\,Q_{r\theta}\Big]P \\ &+ \frac{\epsilon}{2}\Big[\partial^2_{rr}Q_{rr} + 2\partial^2_{r\theta}\frac{Q_{r\theta}}{r} + \partial^2_{\theta\theta}\frac{Q_{\theta\theta}}{r^2}\Big]P \end{aligned} \tag{12}$$

with

$$\begin{aligned} Q_{rr} &= Q_{xx}\cos^2\theta + 2Q_{xy}\sin\theta\cos\theta + Q_{yy}\sin^2\theta \\ Q_{r\theta} &= -Q_{xx}\cos\theta\sin\theta + Q_{xy}(\cos^2\theta - \sin^2\theta) + Q_{yy}\sin\theta\cos\theta \\ Q_{\theta\theta} &= Q_{xx}\sin^2\theta - 2Q_{xy}\sin\theta\cos\theta + Q_{yy}\cos^2\theta \end{aligned} \tag{13}$$

Owing to the nonlinear change of variables, we note that the noise is now process dependent and, as a consequence, there is a noise-dependent term (proportional to ϵ) in the "drift." As we shall see later, the latter term should be taken into account in order to obtain the correct form of the probability density.

In the limit $\epsilon \to 0$, we obtain as a consequence of (3):

$$\lim_{\epsilon \to 0} P(r, \theta; t) = \delta(r - \bar{r}(t))\delta\big(\theta - \bar{\theta}(t)\big) \tag{14}$$

where $\bar{r}(t)$ and $\bar{\theta}(t)$ are governed by the macroscopic equations [cf. Eqs. (4)]. Here we are interested in the approach to the stationary stae, i.e., in the limit $t \to \infty$. It is clear that beyond the bifurcation point the angle variable $\bar{\theta}(t)$ will be periodic for all times, but the radius $\bar{r}(t)$ may evolve to a constant value $\bar{r}_s$. For instance, this is the case near a Hopf bifurcation point where the limit cycle is approximately circular.[3] We can thus scale the r variable as follows:

$$\begin{aligned} &r = \bar{r}_s + \rho\epsilon^{1-b}, \qquad b < 1 \\ &\bar{r}_s \sim O(\epsilon^{2c}), \qquad c > 0 \end{aligned} \tag{15}$$

As we shall see later, the above scaling will not apply for a "hard" transition leading to a limit cycle, since the amplitude fluctuation appears to be macroscopic (see Section 3).

Both the cases of soft and hard transitions will now be treated in detail with the help of explicit examples.

2. HOPF BIFURCATION: THE POINCARÉ MODEL

Consider the set of Langevin equations

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \beta & \omega \\ -\omega & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{bmatrix} + \epsilon^{1/2}\begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{16}$$

where F_x and F_y represent Gaussian white noises defined by (2). For $\epsilon = 0$, the set of equations (16) reduces to the so-called *Poincaré model*:

$$\frac{d}{dt}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}=\begin{bmatrix}\beta & \omega\\ -\omega & \beta\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}-\begin{bmatrix}\bar{x}(\bar{x}^2+\bar{y}^2)\\ \bar{y}(\bar{x}^2+\bar{y}^2)\end{bmatrix} \tag{17}$$

The unique stationary state is given by

$$\bar{x}_s=\bar{y}_s=0 \tag{18}$$

which is stable for $\beta<0$. At $\beta=0$, a Hopf bifurcation occurs. We will be mainly interested in the behavior of the fluctuations close to the bifurcation point, in the limit $\epsilon\to 0$.

In terms of the polar coordinates (11), the Fokker–Planck equation corresponding to (16) reads

$$\partial_t P(r,\theta;t)=-\partial_r\left(\beta r-r^3+\frac{\epsilon}{2r}Q_{\theta\theta}\right)P-\partial_\theta\left(-\omega-\frac{\epsilon}{r^2}Q_{r\theta}\right)P$$
$$+\frac{\epsilon}{2}\left(\partial^2_{rr}Q_{rr}+2\partial^2_{r\theta}\frac{Q_{r\theta}}{r}+\partial^2_\theta\frac{Q_{\theta\theta}}{r^2}\right)P \tag{19}$$

where the definitions (13) have been used. Clearly, in the limit $\epsilon\to 0$, the probability density $P(r,\theta;t)$ reduces to a Dirac δ function centered around $\bar{r}(t)$ and $\bar{\theta}(t)$ with [cf. Eq. (14)]

$$\begin{aligned}\frac{d}{dt}\bar{r}&=\beta\bar{r}-\bar{r}^3\\ \frac{d}{dt}\bar{\theta}&=-\omega\end{aligned} \tag{20}$$

if initially so. For $\beta\geqslant 0$, the latter set of equations admits an orbitally stable periodic solution of circular form with amplitude $\bar{r}_s=\sqrt{\beta}$. Following the discussion in the previous section, we set

$$\begin{aligned}r&=\bar{r}_s+\rho\epsilon^{1-b}\\ \theta&=\theta,\qquad b<1\end{aligned} \tag{21}$$

For $\beta<0$, we obtain $\bar{r}_s=0$ and $b=1/2$. Hence the first-order correction to the result (14) is a bivariate Gaussian distribution, as was already discussed in the previous paper[1] (see also Tomita *et al.*[8]).

We now consider the case $\beta\geqslant 0$. Let us first study the close vicinity of the bifurcation point by setting

$$\beta=\tilde{\beta}\epsilon^{2c},\qquad c>0 \tag{22}$$

where $\tilde{\beta}$ is positive and independent of ϵ. Taking into account (21) and (22) we can rewrite Eq. (19) as follows:

$$\partial_t \mathcal{P}(\rho,\theta;t) = -\partial_\rho \Bigg[-2\tilde{\beta}\rho\epsilon^{2c} - 3\tilde{\beta}^{1/2}\rho^2\epsilon^{1-b+c} - \rho^3\epsilon^{2(1-b)} + \frac{\epsilon^{-1+2b}Q_{\theta\theta}}{2(\rho+\tilde{\beta}^{1/2}\epsilon^{-1+b+c})}\Bigg]\mathcal{P}$$
$$-\partial_\theta\Bigg[-\omega - \frac{\epsilon^{-1+2b}Q_{r\theta}}{(\tilde{\beta}^{1/2}\epsilon^{b-1+c}+\rho)^2}\Bigg]\mathcal{P}$$
$$+\frac{\epsilon^{-1+2b}}{2}\Bigg[\partial^2_{\rho\rho}Q_{rr} + \partial^2_{\rho\theta}\frac{Q_{r\theta}}{(\tilde{\beta}\epsilon^{b-1+c}+\rho)} + \partial^2_\theta \frac{Q_{\theta\theta}}{(\tilde{\beta}^{1/2}\epsilon^{b-1+c}+\rho)^2}\Bigg]\mathcal{P} \tag{23}$$

The problem is now to find the relevant scaling which will lead to a nontrivial limit of the probability density of the scaled variables. Firstly, we can show that necessarily $b > 1/2$. For this purpose, let us first consider the close vicinity of the bifurcation point for which the inequality $c > 1 - b$ holds. Supposing $b \leqslant 1/2$, we obtain to dominant order in ϵ

$$\partial_t\langle\rho^2\rangle = \epsilon^{-1+2b}(Q_{xx} + Q_{yy}) > 0 \tag{24}$$

which shows that the fluctuations of the radial variable diverge in the long time limit. Hence, close to the bifurcation point, the asymptotic time regime should be described by a value $b > 1/2$. It is easy to verify on Eq. (12) together with (15) that this is a general result: in the close vicinity of a Hopf bifurcation we always find

$$b > 1/2 \tag{25}$$

Using now (25) and integrating (23) over θ we obtain

$$\partial_t\mathcal{P}(\rho;t) = O(\epsilon^{-1+2b}, \epsilon^{2(1-b)}, \epsilon^{2c}) \sim o(1) \tag{26}$$

Therefore

$$\partial_t\mathcal{P}(\theta\,|\,\rho;t) = \frac{\partial}{\partial\theta}\omega\mathcal{P}(\theta\,|\,\rho;t) + o(1) \tag{27}$$

where $\mathcal{P}(\theta\,|\,\rho;t)$ represents the conditional probability of θ for a given value of ρ. In the limit $\epsilon \to 0$ the solution of (27) reads

$$\lim_{\epsilon \to 0} \mathcal{P}(\theta \,|\, \rho; t) = \delta(\theta + \omega t) \tag{28}$$

for all finite times, if initially so. On the other hand, it is clear that for all $\epsilon \neq 0$, the probability density $\mathcal{P}(\rho, \theta; t)$ approaches a stationary state in the long time limit. Therefore, taking the limit $t \to \infty$ prior to the limit $\epsilon \to 0$, we obtain

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} \mathcal{P}(\theta \,|\, \rho; t) = \lim_{\epsilon \to 0} \mathcal{P}_{\mathrm{st}}(\theta \,|\, \rho) = \frac{1}{2\pi} \tag{29}$$

Integrating (23) over θ and calculating the conditional averages using (29), we find

$$\begin{aligned} \partial_t \mathcal{P}(\rho; t) = -\partial_\rho \Bigg[& -2\tilde{\beta}\rho\epsilon^{2c} - 3\tilde{\beta}^{1/2}\rho^2\epsilon^{1-b+c} - \rho^3\epsilon^{2(1-b)} \\ & + \frac{\epsilon^{-1+2b} Q}{2(\tilde{\beta}^{1/2}\epsilon^{b+c-1} + \rho)} \Bigg] \mathcal{P} \\ & + \frac{\epsilon^{-1+2b}}{2} Q \partial^2_{\rho\rho} \mathcal{P} \end{aligned} \tag{30}$$

with

$$Q = \tfrac{1}{2}(Q_{xx} + Q_{yy}) \tag{31}$$

The value of b follows from the requirement that drift and diffusion term should equally contribute to the behavior of $\mathcal{P}(\rho; t)$ in the limit $\epsilon \to 0$, and this whatever the value of c. Hence

$$b = 3/4 \tag{32}$$

It follows that the value

$$c = 1/4 \tag{33}$$

delimitates the critical and the Gaussian regime. Taking into account (32) and (33) we finally obtain

$$\partial_\tau \mathcal{P}(\rho; \tau) = -\partial_\rho \left[-2\tilde{\beta}\rho - 3\tilde{\beta}^{1/2}\rho^2 - \rho^3 + \frac{Q}{2(\tilde{\beta}^{1/2} + \rho)} \right] \mathcal{P} + \frac{Q}{2} \partial^2_\rho \mathcal{P} \tag{34}$$

with

$$\tau = t\epsilon^{1/2} \tag{35}$$

We thus obtained a one-variable problem which can be treated by the known classical methods (note, however, the slow time scale τ which is associated to this variable). In particular, at the stationary state we find in terms of the original variables

$$\begin{aligned} \mathcal{P}_{st}(r) &\sim r \exp\left[\frac{2\epsilon^{-1}}{Q}\left(\beta\frac{r^2}{2} - \frac{r^4}{4}\right)\right] \\ \mathcal{P}_{st}(\theta\,|\,r) &= \frac{1}{2\pi} \end{aligned} \tag{36}$$

Note that the crater form of the probability density has circular symmetry (see also Ref. 4).

The above derivation of the stationary solution was done for $\beta \sim O(\epsilon^{1/2})$. However, one readily verifies that it satisfies (30) at the stationary state, whatever the value of β (i.e., $c = 0$). This is a consequence of the limit cycle for the Poincaré model being circular. In more complicated cases, such as the Brusselator,[5] the limit cycle is circular (to within a linear transformation of the variables) only for values $\beta \sim O(\epsilon^{1/2})$ and a result of the type of (36) only applies for this range of the bifurcation parameter. Note finally that for negative values of β, the quartic term in (36) is negligible and the radial distribution is Gaussian.

3. HARD TRANSITION TO THE LIMIT CYCLE: THE VAN DER POL OSCILLATOR

We consider the Van der Pol equation:

$$\frac{d^2}{dt^2}\bar{x} + \beta(\bar{x}^2 - 1)\frac{d}{dt}\bar{x} + \omega^2\bar{x} = 0 \tag{37}$$

where β plays the role of the control parameter. For $\beta < 0$, there exists a unique stable stationary state $\bar{x}_s = 0$. At $\beta = 0$, (37) reduces to the equation of the linear harmonic oscillator. For $\beta > 0$, a stable limit cycle is found. Its radius tends to the finite value 2 for $\beta \to +0$. Hence, there is at the crossing of $\beta = 0$ an abrupt change of the stationary solution from the state $\bar{x}_s = 0$ to a limit cycle behavior with finite radius 2. In order to investigate the effect of fluctuations, we rewrite (37) as a set of two coupled Langevin equations:

$$\begin{aligned} \frac{d}{dt}x &= \omega y + \epsilon^{1/2}F_x \\ \frac{d}{dt}y &= -\omega x + \beta(1 - x^2)y + \epsilon^{1/2}F_y \end{aligned} \tag{38}$$

Note that for $\epsilon = 0$, (38) is equivalent to (37). Furthermore, adding a random force term in the right-hand side of (37), corresponds to the case

$F_x = 0$. As in the previous section, we derive from (38) the corresponding Fokker–Planck equation in polar coordinates:

$$\begin{aligned}\partial_t P(r,\theta;t) = & -\partial_r\left[\beta(1-r^2\cos^2\theta)r\sin^2\theta + \frac{\epsilon}{2r}Q_{\theta\theta}\right]P \\ & -\partial_\theta\left[-\omega+\beta(1-r^2\cos^2\theta)\sin\theta\cos\theta - \frac{\epsilon}{r^2}Q_{r\theta}\right]P \\ & +\frac{\epsilon}{2}\left(\partial^2_{rr}Q_{rr} + 2\partial^2_{r\theta}\frac{Q_{r\theta}}{r} + \partial^2_\theta\frac{Q_{\theta\theta}}{r^2}\right)P \end{aligned} \tag{39}$$

where the definitions (13) have been used. For $\beta < 0$, the probability density is Gaussian. In the limit $\beta \to -0$, the process reduces to the stochastically driven linear harmonic oscillator for which, as well known, no stationary distribution exists. Henceforth, we consider explicitly the limit $\beta \to +0$, at the stationary state. As shown in the previous section, we cannot scale the angle variable θ owing to its time periodic behavior. On the other hand, a scaling of the radial variable r, analogous to (21), proved also to be impossible. Nevertheless, let us investigate the vicinity of the transition point by setting

$$\beta = \tilde{\beta}\epsilon^{2c} \tag{40}$$

It then follows from (39) that for $c > 1/2$, the linear harmonic oscillator is recovered. At $c = 1/2$, the noise and the drift terms have the same order of magnitude. Hereafter, we will be interested only in this case. In fact, for $c < 1/2$ the system exhibits a noncircular limit cycle whose study goes beyond the scope of this paper.

Following the same line as in the previous section, we verify that

$$\frac{\partial}{\partial t}P(r) \sim O(\epsilon) \tag{41}$$

and that

$$\lim_{\epsilon\to 0} P_{\rm st}(\theta\,|\,r) = \frac{1}{2\pi} \tag{42}$$

Inserting this result in (39) and integrating over θ, we obtain to dominant order in ϵ

$$\partial_\tau P(r;\tau) = -\partial_r\left[\frac{\tilde{\beta}}{2}\left(1-\frac{r^2}{4}\right)r + \frac{Q}{2r}\right]P + \frac{Q}{2}\partial^2_{rr}P \tag{43}$$

where we have set $\tau = t\epsilon$ and used the definition (31). At the stationary state, it follows

$$P_{\rm st}(r) \sim r\exp\left\{-\frac{\epsilon^{-1}}{Q}\beta\left[\left(\frac{r}{2}\right)^2 - 1\right]^2\right\} \tag{44}$$

This solution is valid for $\beta \sim O(\epsilon)$. The probability density displays, in the limit $\beta \to +0$, a craterlike form with radius 2. This behavior is to be distinguished from that obtained for the Poincaré model where the radius of the probability crater goes to zero as $\beta^{1/2}$. Moreover, it is clear that the result (44) applies only for positive values of β, while for the Poincaré model the solution (36) is valid for all values of β.

Note finally that the fluctuations for the r variable are of the order $O(1)$. This result shows that, sufficiently close to the transition point, the fluctuations remain macroscopically large, even in the weak noise limit $\epsilon \to 0$. This is not surprising since for the stochastically driven linear harmonic oscillator, to which the system reduces near the transition point ($c > 1/2$), the amplitude fluctuations become infinitely large in the long time limit $t \to \infty$.

4. CONCLUSIONS

In this and the previous paper, we have introduced a method which allows to solve coupled Langevin equations in the limit of weak noise. The introduction of appropriately scaled variables and parameters, and the evaluation of the orders of magnitude of the different terms in the corresponding Fokker–Planck equation, leads to an asymptotic solution of the problem. We have been concerned primarily with the long time limit, describing the approach of the stationary state, in the close vicinity of the bifurcation point. A diversity of behavior was found according to the type of bifurcation (cusp, Hopf, or hard transition) and the specific nonlinearity of the system. On the other hand, a general treatment of the fluctuations in the range of multiple macroscopic steady state has not been achieved with our approach.

Throughout these papers, we considered Langevin equations with process independent noise. A more fundamental description of internal thermodynamic fluctuations is provided by the "master equation." It has the advantage of describing simultaneously the macroscopic dynamics and the corresponding fluctuations coming from the discrete character of the underlying microscopic processes. It has been shown that these two approaches are equivalent in the thermodynamic limit, provided that the macroscopic steady state is unique and globally stable.[6] On the other hand, the status of a Langevin type of approach, for internal thermodynamic fluctuations, beyond the bifurcation point remains an open question.

So far, we considered only homogeneous systems, disregarding spatial fluctuations. The latter can, however, be crucial, for instance, for the formation of inhomogeneous macroscopic steady states (cf. reaction–diffusion system). Another important question in this respect is the role of inhomogeneous fluctuations in the emergence of coherent temporal structures (chemical clocks). It is clear that in these problems the dimensionality of the system will play an important role.[7] We will come back to these problems in a future publication.

ACKNOWLEDGMENTS

We would like to express our gratitude to Professor G. Nicolis and Dr. J. W. Turner for critical discussions and constructive comments on the manuscript. This work has been partially supported by the Belgian Government: A.R.C., Convention No. 76/81.II.3 and the U.S. Department of Energy, Subcontract No. UT-10947.

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